Sharp Nash Inequalities on the unit sphere
The influence of symmetries

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Abstract: In this paper both we establish the best constants for the Nash inequalities on the standard unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ and we give answers on the existence of extremal functions on the corresponding problems. Also we study the problem of the best constants in the case, where the data are invariant under the action of the group $G = O(k) \times O(m)$, and we find the best constants.

Keywords: Manifolds without boundary; Standard unit sphere; Nash inequalities; Best constants; Extremal functions; Symmetries.

1   Introduction

Nash inequalities after their first appearance in the celebrated paper of Nash [14], reappear in some subsequent papers. Specifically, we refer to [2, 1, 6, 9] for manifolds without boundary and [10, 11, 3, 4] for manifolds with boundary. In this paper we are focusing our interest in the special case when the manifold is the standard unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$.

Let (M, g) be a smooth, complete $n-$dimensional Riemannian manifold of infinite volume, where $n \geq 1$.

We say that the Nash inequality (1) is valid if there exists a constant $A > 0$

1

such that for all $u \in C_0^\infty (M)$,

$$\left(\int_M u^2 dV_g \right)^{1+\frac{2}{n}} \leqslant A \int_M |\nabla u|_g^2 dV_g \left(\int_M |u| \, dV_g \right)^{\frac{4}{n}} \tag{1}$$

Such an inequality, as refereed above, first appeared in the celebrated paper of Nash [14], when discussing the Hölder regularity of solutions of divergence form uniformly elliptic equations.

Let $A_0(n)$ be the best constant in Nash's inequality (1) above for the Euclidean space. That is

$$A_0 (n)^{-1} = \inf \left\{ \left. \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \left(\int_{\mathbb{R}^n} |u| \, dx \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u^2 dx \right)^{1+\frac{2}{n}}} \right| u \in C_0^\infty (\mathbb{R}^n), u \not\equiv 0 \right\}$$

This best constant has been computed by Carlen and Loss in [2], together with the characterization of the extremals for the corresponding optimal inequality, as

$$A_0 (n) = \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_{1,n} |\mathcal{B}^n|^{\frac{2}{n}}},$$

where $|\mathcal{B}^n|$ denotes the euclidian volume of the unit ball $\mathcal{B}^n$ in $\mathbb{R}^n$ and $\lambda_{1,n}$ is the first Neumann eigenvalue for the Laplacian for radial functions in the unit ball $\mathcal{B}^n$.

For an example of application of the Nash inequality with the best constant, we refer to Kato [13] and for a geometric proof with an asymptotically sharp constant, we refer to Beckner [1].

For compact Riemannian manifolds, or smooth bounded domains, (see Nirenberg [15]), the Nash inequality still holds with an additional L^1-term and that is why we will refer to this as the L^1-Nash inequality.

Given (M, g) a smooth compact $n-$dimensional Riemannian manifold, $n \geq 2$, we are looking for the existence of real constants A and B such that for any $u \in C^\infty(M)$,

$$\left(\int_M u^2 dV_g \right)^{1+\frac{2}{n}} \leqslant A \int_M |\nabla u|_g^2 dV_g \left(\int_M |u| \, dV_g \right)^{\frac{4}{n}} + B \left(\int_M |u| \, dV_g \right)^{2+\frac{4}{n}} \tag{2}$$

One can define

$$A_{opt}^1(M) = \inf \left\{ A > 0 : \exists \, B > 0 \text{ s.t. } (2) \text{ is true } \forall \, u \in C^\infty (M) \right\}$$

and

$$B_{opt}^1(M) = \inf\left\{B > 0 : \exists\, A > 0 \text{ s.t. } (2) \text{ is true } \forall\, u \in C^\infty(M)\right\}$$

Druet, Hebey and Vaugon proved in [6] that $A_{opt}^1(M) = A_0(n)$, and (2) with its optimal constant $A = A_0(n)$ is sometimes valid and sometimes not, depending on the geometry, specifically on the sign of the curvature. This is another illustration of the important idea of Druet [5] that an inequality may be at the same time localisable and affected by the geometry. On the contrary, $B_{opt}^1(M) = Vol(M)^{-1-2/n}$, where $Vol(M)$ is the volume of the manifold, and (2) with its optimal constant $B_{opt}^1(M) = Vol(M)^{-1-2/n}$ is always valid with geometry playing no role (see also [6]).
For all $u \in C^\infty(M)$, consider now the L^2-Nash inequality

$$\left(\int_M u^2 dV_g\right)^{1+\frac{2}{n}} \leqslant \left(A \int_M |\nabla u|_g^2 dV_g + B \int_M u^2 dV_g\right)\left(\int_M |u|\, dV_g\right)^{\frac{4}{n}} \quad (3)$$

and define

$$A_{opt}^2(M) = \inf\left\{A > 0 : \exists\, B > 0 \text{ s.t. } (3) \text{ is true} \forall\, u \in C^\infty(M)\right\}$$

and

$$B_{opt}^2(M) = \inf\left\{B > 0 : \exists\, A > 0 \text{ s.t. } (3) \text{ is true } \forall\, u \in C^\infty(M)\right\}$$

Humbert studied in [9] the L^2-Nash inequality in detail. Contrary to the sharp L^1-Nash inequality, he proved in this case that B always exists and $A_{opt}^2(M) = A_0(n)$. Also, he studied the second optimal constant $B_{opt}^2(M)$ of this inequality, giving its explicit value $B_{opt}^2(\mathbb{S}^1) = (2\pi)^{-2}$ for $n = 1$ (i.e. for $M = \mathbb{S}^1$), and, for $n > 1$, proving that

$$B_{opt}^2 \geq \max\left(Vol(M)^{-2/n}, \frac{|\mathcal{B}|^{-2/n}}{6n}\left(\frac{2}{n+2} + \frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{2/n}\max_{x\in M} S_g(x)\right),$$

where $|\mathcal{B}|$ is the volume of the unit ball $\mathcal{B}$ in $\mathbb{R}^n$, λ_1 is the first non-zero Neumann eigenvalue of the Laplacian on radial functions on $\mathcal{B}$, $Vol(M)$ is the volume of (M,g) and $S_g(x)$ is the scalar curvature of g at x. In the same paper it was proved that, if (M,g) is a smooth compact Riemaniann $n-$manifold with $n \geq 1$ and L_1-Nash inequality is true, with $A = A_0(n)$

3

and some B, then there exists $u_0 \in H_1(M)$, $u_0 \not\equiv 0$, (where $H_1(M)$ is the standard Sobolev space consisting of functions in L^2 with gradient in L^2), an extremal function for the sharp L^2-Nash inequality (3), that is, a function such that:

$$\left(\int_M u_0^2 dV_g\right)^{1+\frac{2}{n}} = \left(A_0(n)\int_M |\nabla u_0|_g^2 dV_g + B_{opt}(M)\int_M u_0^2 dV_g\right)\left(\int_M |u_0|\, dV_g\right)^{\frac{4}{n}}$$

In this paper we are focusing our interest in the special case where the manifold is the standard unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. We study both Nash's inequalities L^1 and L^2 first in the general case and second in the presence of symmetries.

More precisely:

• We give the proof of the problem of finding the first constant in the L^2-Nash inequality in $\mathbb{S}^n$ and we compute the exact value of the second best constant of this inequality.

• We answer the problem of finding both best constants in the L^1-Nash inequality in $\mathbb{S}^n$.

• We prove the existence of extremal functions in L^2 and non existence in L^1-Nash inequalities.

• We study the problem of the best constants in the L^2-Nash inequality in $\mathbb{S}^n$, $n \geq 3$, where the data are $G-$invariant under the action of the group $G = O(k) \times O(m)$, $k + m = n + 1$, $k \geq m \geq 2$ and we find the best constants in this case.

2 Statement of results

Theorem 2.1 *For all $\phi \in H_1(\mathbb{S}^n)$, $n \geq 1$, there exists a constant B such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant \left(A_0(n)\int_{\mathbb{S}^n} |\nabla\phi|^2\, ds + B\int_{\mathbb{S}^n} \phi^2 ds\right)\left(\int_{\mathbb{S}^n} |\phi|ds\right)^{\frac{4}{n}} \quad (4)$$

Moreover the constant $A_0(n)$ is the optimal for this inequality.

Theorem 2.2 *For all $\phi \in H_1(\mathbb{S}^n)$, $n \geq 1$, there exists a constant A such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant \left(A\int_{\mathbb{S}^n} |\nabla\phi|^2 ds + \omega_n^{-\frac{2}{n}}\int_{\mathbb{S}^n} \phi^2 ds\right)\left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{4}{n}}, \quad (5)$$

where ω_n denotes the volume of the standard unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. In particular

$$\omega_{2n} = \frac{(4\pi)^n (n-1)!}{(2n-1)!} \quad \text{and} \quad \omega_{2n+1} = \frac{2\pi^{n+1}}{n!}$$

Moreover $\omega_n^{-\frac{2}{n}}$ is the optimal constant for this inequality.
In addition there exists $\phi_0 \in H_1(\mathbb{S}^n)$, $\phi_0 \not\equiv 0$, an extremal function for the sharp L^2-inequality $(N(A_0(n), \omega_n^{-\frac{2}{n}})$, that is, such that

$$\left(\int_{\mathbb{S}^n} \phi_0^2 ds\right)^{1+\frac{2}{n}} = \left(A_0(n)\int_{\mathbb{S}^n} |\nabla\phi_0|^2 ds + \omega_n^{-\frac{2}{n}}\int_{\mathbb{S}^n} \phi_0^2 ds\right)\left(\int_{\mathbb{S}^n} |\phi_0| ds\right)^{\frac{4}{n}} \quad (6)$$

Theorem 2.3 *For all $\phi \in H_1(\mathbb{S}^n)$ there exists a constant B_ε such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant (A_0(n) + \varepsilon)\int_{\mathbb{S}^n} |\nabla\phi|^2 ds \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{4}{n}} + B_\varepsilon \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{2+\frac{4}{n}} \quad (7)$$

Moreover the constant $A_0(n)$ is the optimal for this inequality.

Theorem 2.4 *For all $\phi \in H_1(\mathbb{S}^n)$ there exists a constant A such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant A\int_{\mathbb{S}^n} |\nabla\phi|^2 ds \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{4}{n}} + \omega_n^{-1-\frac{2}{n}} \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{2+\frac{4}{n}} \quad (8)$$

Moreover $\omega_n^{-1-\frac{2}{n}}$ is the optimal constant for this inequality.

Corollary 2.1 *The inequality of Theorem 2.3 is false if $\varepsilon = 0$.*

Corollary 2.2 *There do not exist extremal functions for the sharp L^1-Nash inequality $N(A_0(n), B^1_{opt})$.*

Theorem 2.5 *For all $f \in H_{1,G}(\mathbb{S}^n)$, $n \geq 3$, there exists a constant B such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} f^2 ds \right)^{1+\frac{2}{k}} \leqslant \left(A_0(k)\omega_{n-k}^{-\frac{2}{k}} \int_{\mathbb{S}^n} |\nabla f|^2 ds + B \int_{\mathbb{S}^n} f^2 ds \right) \left(\int_{\mathbb{S}^n} |f| ds \right)^{\frac{4}{k}} \quad (9)$$

Moreover the constant $A_0(k)\omega_{n-k}^{-\frac{2}{k}}$ is the optimal for this inequality.

Theorem 2.6 *For all $f \in H_{1,G}(\mathbb{S}^n)$, $n \geq 3$, there exists a constant A such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} f^2 ds \right)^{1+\frac{2}{k}} \leqslant \left(A \int_{\mathbb{S}^n} |\nabla f|^2 ds + \omega_n^{-\frac{2}{k}} \int_{\mathbb{S}^n} f^2 ds \right) \left(\int_{\mathbb{S}^n} |f| ds \right)^{\frac{4}{k}} \quad (10)$$

Moreover the constant $\omega_n^{-\frac{2}{k}}$ is the optimal for this inequality.

3   Notations and preliminary results

3.1   The General Case

Consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, of dimension n and radius 1. That is

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

The stereographic projection

$$\Pi : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$$

maps a point $P' \in \mathbb{S}^n \backslash \{N\}$ into the intersection $P \in \mathbb{R}^n$ of the line joining P' and the north pole $N = (0, 0, ..., 1)$ with $\mathbb{R}^n$.
Let $g_{\alpha\beta}$ the standard metric of $\mathbb{S}^n$ (i.e. the one inherited from $\mathbb{R}^{n+1}$) is expressed in terms of stereographic coordinates by

$$g_{\alpha\beta} = \left(\frac{2}{1+|x|^2} \right)^2 \delta_{\alpha\beta}.$$

Hence the standard volume element of $\mathbb{S}^n$ is

$$ds = \left(\frac{2}{1+|x|^2} \right)^n dx$$

6

Let $H_1(\mathbb{S}^n)$ be the standard Sobolev space consisting of functions in $L^2(\mathbb{S}^n)$ with gradient in $L^2(\mathbb{S}^n)$. For any function $\phi \in H_1(\mathbb{S}^n)$ set $u = \phi \circ \Pi^{-1}$. The integral and the gradient Dirichlet integral, corresponding to a conformal metric $ds = p^n dx$, where $p = \frac{2}{1+|x|^2}$, are:

$$\int_{\mathbb{S}^n} \phi\, ds = \int_{\mathbb{R}^n} u\, p^n dx \tag{11}$$

$$\int_{\mathbb{S}^n} |\nabla \phi|^2 ds = \int_{\mathbb{R}^n} |\nabla u|^2 p^{n-2} dx \tag{12}$$

We may assume that $\mathbb{S}^n$ is covered by a finite number of charts, say $(U_j, \xi_j), 1 \leq j \leq N$, such for any $\varepsilon > 0$, (U_j, ξ_j) can be chosen such that:

$$1 - \varepsilon \leq \sqrt{\det(g^j_{\alpha\beta})} \leq 1 + \varepsilon \quad \text{on} \quad U_j, \quad \text{for} \quad 1 \leq \alpha, \beta \leq n \tag{13}$$

where the $g^j_{\alpha\beta}$'s are the components of g in (U_j, ξ_j).
For each j we consider $h_j \in C_0^\infty(\mathbb{R}^n)$, $h_j \geq 0$ and set

$$\eta_j = \frac{h_j \circ \xi_j}{\sum_{j=1}^N (h_j \circ \xi_j)} \tag{14}$$

The η_j's are then a partition of unity for $\mathbb{S}^n$ relative to U_j's.

Lemma 3.1 *For any $\varepsilon > 0$ and for all $\phi \in C_0^\infty(\mathbb{S}^n)$ the following inequality holds*

$$\left(\int_{\mathbb{S}^n} (\eta_j \phi)^2 \, ds \right)^{1+\frac{2}{n}} \leqslant (A_0(n) + \varepsilon) \int_{\mathbb{S}^n} |\nabla(\eta_j \phi)|^2 ds \left(\int_{\mathbb{S}^n} |\eta_j \phi| \, ds \right)^{\frac{4}{n}} \tag{15}$$

Proof. By (11) and (12) because of (13) for any $\phi \in C_0^\infty(\mathbb{S}^n)$ and any $q \geq 1$ real, setting $(\eta_j \phi) \circ \Pi^{-1} = u_j$ we obtain

$$(1 - \varepsilon)^n \int_{\mathbb{R}^n} (u_j)^q \, dx \leq \int_{\mathbb{S}^n} (\eta_j \phi)^q \, ds \leqslant (1 + \varepsilon)^n \int_{\mathbb{R}^n} (u_j)^q \, dx \tag{16}$$

and

$$(1 - \varepsilon)^{n-2} \int_{\mathbb{R}^n} |\nabla u_j|^2 \, dx \leq \int_{\mathbb{S}^n} |\nabla(\eta_j \phi)|^2 \, ds \leqslant (1 + \varepsilon)^{n-2} \int_{\mathbb{R}^n} |\nabla u_j|^2 \, dx \tag{17}$$

7

It is known, by Carlen and Loss [2], that for any $u \in C_0^\infty(\mathbb{R}^n)$, the following inequality holds

$$\left(\int_{\mathbb{R}^n} u^2 dx \right)^{1+\frac{2}{n}} \leqslant A_0(n) \int_{\mathbb{R}^n} |\nabla u|^2 dx \left(\int_{\mathbb{R}^n} |u| \, dx \right)^{\frac{4}{n}}$$

For any $\varepsilon > 0$, we can choose $\delta > 0$ such that for any $x = \xi_j(s) \in \mathbb{R}^n, s \in U_j \subset \mathbb{S}^n$ and for all $u \in C_0^\infty(B_x(\delta))$, $(B_x(\delta) \subset \xi_j(U_j)$ is the $n-$dimensional ball of radius δ centered on x), the following inequality holds

$$\left(\int_{\mathbb{R}^n} u^2 dx \right)^{1+\frac{2}{n}} \leqslant \left(A_0(n) + \frac{\varepsilon}{2} \right) \int_{\mathbb{R}^n} |\nabla u|^2 dx \left(\int_{\mathbb{R}^n} |u| \, dx \right)^{\frac{4}{n}} \qquad (18)$$

From (17) because of (15) and (16) we obtain

$$
\begin{aligned}
\left(\int_{\mathbb{S}^n} (\eta_j \phi)^2 \, ds \right)^{1+\frac{2}{n}} &\leqslant (1+\varepsilon)^{n+2} \left(\int_{\mathbb{R}^n} (u_j)^2 \, dx \right)^{1+\frac{2}{n}} \\
&\leqslant (1+\varepsilon)^{n+2} A_0(n) \int_{\mathbb{R}^n} |\nabla u_j|^2 dx \left(\int_{\mathbb{R}^n} |u_j| \, dx \right)^{\frac{4}{n}} \\
&\leqslant (1+\varepsilon)^{n+2} \frac{1}{(1-\varepsilon)^{n-2}} \frac{1}{(1+\varepsilon)^4} \\
&\quad \times \left(A_0(n) + \frac{\varepsilon}{2} \right) \int_{\mathbb{S}^n} |\nabla(\eta_j \phi)|^2 ds \left(\int_{\mathbb{S}^n} |\eta_j \phi| \, ds \right)^{\frac{4}{n}} \\
&= \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^{n-2} \left(A_0(n) + \frac{\varepsilon}{2} \right) \int_{\mathbb{S}^n} |\nabla(\eta_j \phi)|^2 ds \\
&\quad \times \left(\int_{\mathbb{S}^n} |\eta_j \phi| \, ds \right)^{\frac{4}{n}} \qquad (19)
\end{aligned}
$$

Since the function $f : (0,1) \to (1, +\infty)$ with $f(\varepsilon) = \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^{n-2}$ is monotonically increasing, we can choose the $\varepsilon > 0$ such that the inequality

$$f(\varepsilon) \left(A_0(n) + \frac{\varepsilon}{2} \right) \leq A_0(n) + \varepsilon$$

holds. Hence from (19) follows (15) and the lemma is proved. $\qquad \square$

8

Lemma 3.2 *For any $\varepsilon > 0$ and for all $\phi \in C_0^\infty(\mathbb{S}^n)$ there exists a constant $B_\varepsilon > 0$ such that the following inequality holds*

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant (A_0(n) + \varepsilon) \left(\int_{\mathbb{S}^n} |\nabla \phi|^2 ds + B_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds\right) \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{4}{n}} (20)$$

Proof. We set $\alpha_j = \frac{\eta_j^2}{\sum_{m=1}^N \eta_m^2}, j = 1, 2, .., N$, where η_j is defined by (14), and so $\{\alpha_j\}_{j=1,2,...N}$ is a partition of unity for $\mathbb{S}^n$ subordinated in the covering $(U_j)_{j=1,2,...,N}$, functions $\sqrt{\alpha_j}$ are smooth and there exist a positive constant H such that for any $j = 1, ..., N$ holds

$$|\nabla \sqrt{\alpha_j}| \leqslant H \tag{21}$$

Let $\phi \in C^\infty(\mathbb{S}^n)$. Then we have

$$\|\phi\|_2^2 = \|\phi^2\|_1 = \|\sum_{j=1}^N \alpha_j \phi^2\|_1 \leq \sum_{j=1}^N \|\alpha_j \phi^2\|_1 = \sum_{j=1}^N \|\sqrt{\alpha_j}\phi\|_2^2 \tag{22}$$

By Lemma 3.1, for any j,

$$\|\sqrt{\alpha_j}\phi\|_2^2 \leq A^{\frac{n}{n+2}} \|\nabla(\sqrt{\alpha_j}\phi)\|_2^{\frac{2n}{n+2}} \|\sqrt{\alpha_j}\phi\|_1^{\frac{4}{n+2}} \tag{23}$$

where

$$A = A_0(n) + \varepsilon.$$

By Hölder's inequality,

$$\|\sqrt{\alpha_j}\phi\|_1 \leq \|\alpha_j\phi\|_1^{\frac{1}{2}} \|\phi\|_1^{\frac{1}{2}} \tag{24}$$

As a consequence, by (22), (23) and (24), for any $\phi \in C^\infty(\mathbb{S}^n)$ we obtain

$$\int_{\mathbb{S}^n} \phi^2 ds \;\; \leqslant \;\; A^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{2}{n+2}} \sum_{j=1}^N \left(\int_{\mathbb{S}^n} |\nabla(\sqrt{\alpha_j}\phi)|^2 ds\right)^{\frac{n}{n+2}}$$

$$\times \left(\int_{\mathbb{S}^n} \alpha_j |\phi| ds\right)^{\frac{2}{n+2}} \tag{25}$$

Moreover, for any a_j, b_j non negative and for all $p \geqslant 1, q \geqslant 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, by the Hölder's inequality in the discreet case it holds

$$\sum_{j=1}^N a_j b_j \leqslant \left(\sum_{j=1}^N a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^N b_j^q\right)^{\frac{1}{q}} \tag{26}$$

9

Setting in (43)

$$a_j = \left(\int_{\mathbb{S}^n} |\nabla(\sqrt{\alpha_j}\phi)|^2 ds \right)^{\frac{n}{n+2}}, \quad b_j = \left(\int_{\mathbb{S}^n} \alpha_j |\phi| ds \right)^{\frac{2}{n+2}}$$

$$p = \frac{n+2}{n}, \quad q = \frac{n+2}{2}$$

we obtain

$$\sum_{j=1}^{N} \left(\int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds \right)^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} \alpha_j |\phi| ds \right)^{\frac{2}{n+2}}$$

$$\leqslant \left(\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds \right)^{\frac{n}{n+2}} \left(\sum_{j=1}^{N} \int_{\mathbb{S}^n} \alpha_j |\phi| ds \right)^{\frac{2}{n+2}}$$

$$= \left(\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds \right)^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} \left(\sum_{j=1}^{N} \alpha_j \right) |\phi| ds \right)^{\frac{2}{n+2}}$$

$$= \left(\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds \right)^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{2}{n+2}} \qquad (27)$$

By (42) and (44) we obtain

$$\int_{\mathbb{S}^n} \phi^2 ds \leqslant A^{\frac{n}{n+2}} \left(\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds \right)^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n+2}} \quad (28)$$

Furthermore, since

$$\left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 = \alpha_j |\nabla \phi|^2 + \phi^2 \left| \nabla \left(\sqrt{\alpha_j} \right) \right|^2 + 2 \left\langle \nabla \phi, \nabla \left(\sqrt{\alpha_j} \right) \right\rangle \phi \sqrt{\alpha_j}$$

and because of (21), we obtain

$$\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j}\phi \right) \right|^2 ds = \sum_{j=1}^{N} \int_{\mathbb{S}^n} \alpha_j |\nabla \phi|^2 ds + \sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j} \right) \right|^2 \phi^2 ds$$

$$+ 2 \sum_{j=1}^{N} \int_{\mathbb{S}^n} \left\langle \nabla \left(\sqrt{\alpha_j} \right), \nabla \phi \right\rangle \phi \sqrt{\alpha_j} ds$$

10

$$
= \sum_{j=1}^{N} \int_{\mathbb{S}^n} \alpha_j \left| \nabla \phi \right|^2 ds + \sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j} \right) \right|^2 \phi^2 ds
$$

$$
+ 2 \int_{\mathbb{S}^n} \sum_{j=1}^{N} \left\langle \nabla \left(\sqrt{\alpha_j} \right), \nabla \phi \right\rangle \phi \sqrt{\alpha_j} ds
$$

$$
= \int_{\mathbb{S}^n} \left(\sum_{j=1}^{N} \alpha_j \right) \left| \nabla \phi \right|^2 ds
$$

$$
+ \int_{\mathbb{S}^n} \sum_{j=1}^{N} \alpha_j \left| \nabla \left(\sqrt{\alpha_j} \right) \right|^2 \phi^2 ds
$$

$$
+ 2 \int_{\mathbb{S}^n} \sum_{j=1}^{N} \left\langle \nabla \left(\sqrt{\alpha_j} \right), \nabla \phi \right\rangle \phi \sqrt{\alpha_j} ds
$$

$$
\leqslant \int_{\mathbb{S}^n} \left| \nabla \phi \right|^2 ds + HN \int_{\mathbb{S}^n} \phi^2 ds
$$

$$
+ 2 \int_{\mathbb{S}^n} \sum_{j=1}^{N} \left\langle \nabla \left(\sqrt{\alpha_j} \right), \nabla \phi \right\rangle \phi \sqrt{\alpha_j} ds
$$

$$
= \int_{\mathbb{S}^n} \left| \nabla \phi \right|^2 ds + C_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds
$$

$$
+ 2 \int_{\mathbb{S}^n} \sum_{j=1}^{N} \left\langle \nabla \left(\sqrt{\alpha_j} \right), \nabla \phi \right\rangle \phi \sqrt{\alpha_j} ds
$$

But

$$
2 \sum_{j=1}^{N} \sqrt{\alpha_j} \nabla \left(\sqrt{\alpha_j} \right) = \sum_{j=1}^{N} 2 \sqrt{\alpha_j} \nabla \left(\sqrt{\alpha_j} \right) = \sum_{j=1}^{N} \left(\nabla \alpha_j \right) = \nabla \left(\sum_{j=1}^{N} \alpha_j \right) = 0
$$

Thus the following inequality is true

$$
\sum_{j=1}^{N} \int_{\mathbb{S}^n} \left| \nabla \left(\sqrt{\alpha_j} \phi \right) \right|^2 ds \leqslant \int_{\mathbb{S}^n} \left| \nabla \phi \right|^2 ds + B_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds \tag{29}
$$

Hence by (45) and (29) we obtain

$$
\int_{\mathbb{S}^n} \phi^2 ds \leqslant A^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} \left| \nabla \phi \right|^2 ds + B_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds \right)^{\frac{n}{n+2}} \left(\int_{\mathbb{S}^n} \left| \phi \right| ds \right)^{\frac{4}{n+2}}
$$

11

or

$$\left(\int_{\mathbb{S}^n} \phi^2 ds\right)^{1+\frac{2}{n}} \leqslant (A_0(n) + \varepsilon) \left(\int_{\mathbb{S}^n} |\nabla\phi|^2 ds + B_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds\right) \left(\int_{\mathbb{S}^n} |\phi| ds\right)^{\frac{4}{n}}$$

and the lemma is proved. □

3.2 The Case of Existence of Symmetries for $n \geq 3$

Let
$$\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^m = \{(x, y) : x \in \mathbb{R}^k, \ y \in \mathbb{R}^m\},$$

where $k + m = n + 1$, $k \geq m \geq 2$.
Then
$$\mathbb{S}^n = \{(x, y) : |x|^2 + |y|^2 = 1\}$$

Let $x = (x^1, x^2, ..., x^k) \in \mathbb{R}^k$ and $y = (x^{k+1}, x^{k+2}, ..., x^{n+1}) \in \mathbb{R}^m$, where $\{x^i, \ i = 1, 2, ..., n+1\}$ is a coordinate system of $\mathbb{R}^{n+1}$.

It is well known that $\mathbb{S}^n$ enjoys a lot of symmetries, namely, the compact Lie group $O(n+1)$ acts isometrically on $\mathbb{S}^n$. Let now $G = O(k) \times O(m)$. Then G is a compact subgroup of $O(n+1)$. For $g = (g_1, g_2) \in G$, where $g_1 \in O(k)$ and $g_2 \in O(m)$, the action of G on S^n is defined by $g(x, y) = (g_1 x, g_2 y)$ and if $P(x, y) \in \mathbb{S}^n$ its orbit under the action of G, since $|x|^2 + |y|^2 = 1$, is

$$O_P = \mathbb{S}^{k-1}(|x|) \times \mathbb{S}^{m-1}(|y|) = \mathbb{S}^{k-1}(|x|) \times \mathbb{S}^{m-1}(\sqrt{1 - |x|^2})$$

Denote $C_G^\infty(\mathbb{S}^n)$ the space of all G−invariant functions under the action of the group G and $H_{1,G}(\mathbb{S}^n)$ the space of all G−invariant functions of $H_1(\mathbb{S}^n)$. Under the above considerations, if $f \in H_{1,G}(\mathbb{S}^n)$, we can set $|x| = sin\theta$ and $|y| = cos\theta$, $0 \leq \theta \leq \pi/2$ and then f is a function of one variable θ and the following formulas hold:

$$\int_{\mathbb{S}^n} |f|^q ds = \omega_{k-1}\omega_{m-1} \int_0^{\pi/2} |f|^q sin^{k-1}\theta \ cos^{m-1}\theta d\theta \qquad (30)$$

$$\int_{\mathbb{S}^n} |\nabla f|^2 ds = \omega_{k-1}\omega_{m-1} \int_0^{\pi/2} (f')^2 sin^{k-1}\theta \ cos^{m-1}\theta d\theta \qquad (31)$$

12

4 Proofs

Proof of Theorem 2.1. For $n = 1$, the theorem is true, see Theorem 2.5 in [9]. Let $n \geq 2$. In order to prove inequality (9) it is equivalence to proving that for all $\phi \in H_1^2(\mathbb{S}^n)$ there exists a constant B' such that the following inequality holds

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1+\frac{2}{n}} \leqslant A_0(n) \left(\int_{\mathbb{S}^n} |\nabla \phi|^2 ds + B' \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}}$$

We use a proof based on Lemma 3.2. Suppose by contradiction that the inequality is not true. Then for any $\alpha > 0$ there exists $\phi_\alpha \in C_0^\infty (\mathbb{S}^n)$ such that

$$\frac{\left(\int_{\mathbb{S}^n} |\nabla \phi_\alpha|^2 ds + \alpha \int_{\mathbb{S}^n} \phi_\alpha^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi_\alpha| ds \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{S}^n} \phi_\alpha^2 ds \right)^{1+\frac{2}{n}}} < \frac{1}{A_0(n)} \tag{32}$$

By (32) because of (11) and (12) we obtain equivalently

$$\frac{\left(\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 p^{n-2} dx + \alpha \int_{\mathbb{R}^n} u_\alpha^2 p^n dx \right) \left(\int_{\mathbb{R}^n} |u_\alpha| p^n dx \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u_\alpha^2 p^n dx \right)^{1+\frac{2}{n}}} < \frac{1}{A_0(n)} \tag{33}$$

where $u_\alpha = \phi_\alpha \circ \Pi^{-1}$.

For any $\lambda > 0$, define u_{α_λ} by $u_{\alpha_\lambda}(x) = u_\alpha(\lambda x)$. So, for any λ, u_{α_λ} has compact support and since $p = \frac{2}{1+|x|^2}$ the following hold

$$\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 p^{n-2} dx = \lambda^{n-2} \int_{\mathbb{R}^n} |\nabla u_{\alpha_\lambda}|^2 \left(\frac{2}{1 + |\frac{x}{\lambda}|^2} \right)^{n-2} dx \tag{34}$$

$$\int_{\mathbb{R}^n} u_\alpha^2 p^n dx = \lambda^n \int_{\mathbb{R}^n} u_{\alpha_\lambda}^2 \left(\frac{2}{1 + |\frac{x}{\lambda}|^2} \right)^n dx \tag{35}$$

$$\left(\int_{\mathbb{R}^n} |u_\alpha| p^n dx \right)^{\frac{4}{n}} = \lambda^4 \left(\int_{\mathbb{R}^n} |u_{\alpha_\lambda}| \left(\frac{2}{1 + |\frac{x}{\lambda}|^2} \right)^n dx \right)^{\frac{4}{n}} \tag{36}$$

$$\left(\int_{\mathbb{R}^n} u_\alpha^2 p^n dx \right)^{1+\frac{2}{n}} = \lambda^{n+2} \left(\int_{\mathbb{R}^n} u_{\alpha_\lambda}^2 \left(\frac{2}{1 + |\frac{x}{\lambda}|^2} \right)^n dx \right)^{1+\frac{2}{n}} \tag{37}$$

13

By (33) because of (34), (35), (36) and (37), for $\lambda \to \infty$, we obtain

$$\frac{\int_{\mathbb{R}^n} |\nabla u_{\alpha_\lambda}|^2 2^{n-2} dx \left(\int_{\mathbb{R}^n} |u_{\alpha_\lambda}|^{2n} dx \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u_{\alpha_\lambda}^2 2^n dx \right)^{1+\frac{2}{n}}} < \frac{1}{A_0(n)}$$

or

$$\frac{\int_{\mathbb{R}^n} |\nabla u_{\alpha_\lambda}|^2 dx \left(\int_{\mathbb{R}^n} |u_{\alpha_\lambda}| dx \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u_{\alpha_\lambda}^2 dx \right)^{1+\frac{2}{n}}} < \frac{1}{A_0(n)}$$

Because of Carlen-Loss Theorem [2], last inequality is false and the theorem is proved. $\qquad\square$

Proof of Theorem 2.2. For $n = 1$, the theorem is true, see Corollary 5.2 in [9].
If $n = 2$, by Theorem 1.2 in [9], we produce that

$$B_{opt}(\mathbb{S}^2) \geq \omega_2^{-1} \tag{38}$$

Let $n \geq 3$. By Theorem 2.1 follows that for all $\phi \in H_1^2(\mathbb{S}^n)$ there exists a constant B such that the following inequality holds

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1+\frac{2}{n}} \leqslant \left(A_0(n) \int_{\mathbb{S}^n} |\nabla \phi|^2 ds + B \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}} \tag{39}$$

On the one hand, by taking $\phi = 1$ in (39), one obtains that $B \geq \omega_n^{-\frac{2}{n}}$. In particular

$$B_{opt}(\mathbb{S}^n) \geq \omega_n^{-\frac{2}{n}} \tag{40}$$

On the other hand, by Hölder's inequality, for any $\phi \in H_1^2(\mathbb{S}^n)$ and for $p = \frac{2n}{n-2}$, it holds that

$$\int_{\mathbb{S}^n} \phi^2 ds \leqslant \left(\int_{\mathbb{S}^n} \phi^p ds \right)^{\frac{1}{p-1}} \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{p-2}{p-1}}$$

or this

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{\frac{2(p-1)}{p}} \leqslant \left(\int_{\mathbb{S}^n} \phi^p ds \right)^{\frac{2}{p}} \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{2(p-2)}{p}} \tag{41}$$

14

By Theorem 4.2 in [7], there exists $A \in \mathbb{R}$ such that for any $\phi \in H_1^2(\mathbb{S}^n)$, holds

$$\left(\int_{\mathbb{S}^n} \phi^p ds \right)^{\frac{2}{p}} \leq A \int_{\mathbb{S}^n} |\nabla \phi|^2 ds + \omega_n^{-\frac{2}{n}} \int_{\mathbb{S}^n} \phi^2 ds \tag{42}$$

By (41), because of (42), we have

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{\frac{2(p-1)}{p}} \leqslant \left(A \int_{\mathbb{S}^n} |\nabla \phi|^2 ds + \omega_n^{-\frac{2}{n}} \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{2(p-2)}{p}}$$

and since

$$\frac{2(p-1)}{p} = 2 - \frac{2}{p} = 2 - \frac{n-2}{n} = 1 + \frac{2}{n},$$

$$\frac{2(p-2)}{p} = 2 - \frac{4}{p} = 2 - \frac{2(n-2)}{n} = \frac{4}{n}$$

we finally obtain

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1 + \frac{2}{n}} \leqslant \left(A \int_{\mathbb{S}^n} |\nabla \phi|^2 ds + \omega_n^{-\frac{2}{n}} \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}}$$

From this inequality and the definition of $B_{opt}(\mathbb{S}^n)$, we obtain

$$\omega_n^{-\frac{2}{n}} \geq B_{opt}(\mathbb{S}^n), \quad \forall \ n \geq 2. \tag{43}$$

Further (38), (40) and (43) yield

$$B_{opt}(\mathbb{S}^n) = \omega_n^{-\frac{2}{n}}, \quad \forall \ n \geq 2$$

For the second part of the Theorem, suppose by contradiction that for all $\phi \in C_0^\infty(\mathbb{S}^n)$ the following inequality holds

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1 + \frac{2}{n}} < A_0(n) \left(\int_{\mathbb{S}^n} |\nabla \phi|^2 ds + A_0^{-1}(n) \omega_n^{-\frac{2}{n}} \int_{\mathbb{S}^n} \phi_0^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}}$$

or

$$\frac{\left(\int_{\mathbb{S}^n} |\nabla \phi|^2 ds + A_0^{-1}(n) \omega_n^{-\frac{2}{n}} \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}}}{\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1 + \frac{2}{n}}} > \frac{1}{A_0(n)}$$

Following the same steps as in the first part of theorem we conclude that for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} u^2 dx \right)^{1+\frac{2}{n}} < A_0(n) \int_{\mathbb{R}^n} |\nabla u|^2 dx \left(\int_{\mathbb{R}^n} |u| dx \right)^{\frac{4}{n}},$$

which is false since, according to [2], there exists an integrable function f_n on $\mathbb{R}^n$, such that its distributional gradient is a square integrable function such that, the equality bellow holds

$$\left(\int_{\mathbb{R}^n} f_n^2 dx \right)^{1+\frac{2}{n}} = A_0(n) \int_{\mathbb{R}^n} |\nabla f_n|^2 dx \left(\int_{\mathbb{R}^n} |f_n| dx \right)^{\frac{4}{n}}.$$

Moreover, it is easy to verify that constant functions are extremal functions for the sharp L^2-Nash inequality and the theorem is proved. $\qquad \square$

Proof of Theorem 2.3. Let

$$a = \int_{\mathbb{S}^n} \phi^2 ds \quad \text{and} \quad b = \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}}$$

Mimicking what is done in [6], let $\varepsilon_1 > 0$ to be chosen later on, and set

$$p = \frac{n+2}{n} \quad \text{and} \quad q = \frac{n+2}{2}$$

Then

$$\frac{1}{p} + \frac{1}{q} = 1$$

and so, by the elementary inequality

$$xy \le \frac{x^p}{p} + \frac{y^q}{q} \quad \text{for all} \quad x, y \ge 0, \quad \text{and for all} \quad p, q \ge 0 \;\; s.t. \;\; \frac{1}{p} + \frac{1}{q} = 1,$$

for $x = a\varepsilon_1$ and $y = \frac{b}{\varepsilon_1}$ we obtain

$$\int_{\mathbb{S}^n} \phi^2 ds \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}} \le \frac{n\varepsilon_1^{\frac{n+2}{n}}}{n+2} \left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1+\frac{2}{n}} + \frac{2\varepsilon_1^{-\frac{n+2}{2}}}{n+2} \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{2+\frac{4}{n}}$$
$$\tag{44}$$

16

By Lemma 3.2 arises that, for any $\varepsilon > 0$ and for all $\phi \in C_0^\infty(\mathbb{S}^n)$ there exists a constant $B_\varepsilon > 0$ such that the following inequality holds

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1+\frac{2}{n}} \leqslant A \left(\int_{\mathbb{S}^n} |\nabla \phi|^2 \, ds + B_\varepsilon \int_{\mathbb{S}^n} \phi^2 ds \right) \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}} \qquad (45)$$

where

$$A = A_0(n) + \frac{\varepsilon}{2}.$$

Combining (44) and (45) we obtain

$$\left(\int_{\mathbb{S}^n} \phi^2 ds \right)^{1+\frac{2}{n}} \leqslant \frac{A}{C} \int_{\mathbb{S}^n} |\nabla \phi|^2 \, ds \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{\frac{4}{n}} + B \left(\int_{\mathbb{S}^n} |\phi| ds \right)^{2+\frac{4}{n}}$$

where

$$C = 1 - \frac{n}{n+2} \varepsilon_1^{\frac{n+2}{n}} A B_\varepsilon \quad \text{and} \quad B = \frac{A B_\varepsilon}{C} \frac{2}{n+2} \varepsilon_1^{-\frac{n+2}{2}}$$

We can choose ε_1 such that

$$\frac{A}{C} = A_0(n) + \varepsilon$$

and the theorem is proved.

Proof of Theorem 2.4 was discussed in [6], (see Theorem 3.1). $\qquad \square$

Proof of Corollary 2.1. Since the scalar curvature of $\mathbb{S}^n$ is $n(n-1) > 0$ our result arises immediately from Theorem 1.3 of [6]. $\qquad \square$

Proof of Corollary 2.1. The conclusion arises immediately by Theorems 2.3 and 2.4. $\qquad \square$

Proof of Theorem 2.5. Let $\varepsilon > 0$ be given. We consider $P \in M$ and its orbit O_P of dimension k. For any $Q = \tau(P) \in O_P$, where $\tau \in G$, we build a chart around Q, denoted by $(\tau(\Omega_P), \xi_P \circ \tau^{-1})$ and "isometric" to (Ω_P, ξ_P). O_P is then covered by such charts. We denote by $(\Omega_m)_{m=1,\dots,M}$ a finite extract covering. We then choose $\delta > 0$ small enough, depending on P and ε, such that $O_{P,\delta} = \{Q \in \mathbb{S}^n : d(Q, O_P) < \delta\}$ the neighborhood $O_{P,\delta}$, (where $d(\cdot, O_P)$ is the distance to the orbit) has the following properties:
(i) $\overline{O_{P,\delta}}$ is a submanifold of $\mathbb{S}^n$ with boundary,

17

(ii) $d^2(\cdot, O_P)$, is a C^∞ function on $O_{P,\delta}$ and

(iii) $O_{P,\delta}$ is covered by $(\Omega_m)_{m=1,\dots,M}$.

Clearly, $\mathbb{S}^n$ is covered by $\cup_{P \in \mathbb{S}^n} O_{P,\delta}$. We denote by $(O_{j,\delta})_{j=1,\dots,J}$ a finite extract covering of $\mathbb{S}^n$, where all $O_{j,\delta}$'s are covered by $(\Omega_{jm})_{m=1,\dots,M_j}$. On each $(O_{j,\delta}), j = 1, \dots, J$ we consider functions depending only on the distance to O_P, and we build a partition of unity (η_j) relative to $O_{j,\delta}$ such that for any j, $\eta_j \in C_G^\infty$. For any $f \in C_G^\infty$, $\eta_j f \in C_G^\infty$ has compact support in $O_{j,\delta}$ and is a function of one variable. Thus this partition of unity corresponds a subdivision of the interval of integration $[0, \pi/2]$ consisted of J subintervals $[\theta_{j-1}, \theta_j]$, not necessarily of equal length.

For any subinterval $[\theta_{j-1}, \theta_j]$ there exists a small $\varepsilon_j > 0$, such that

$$(1 + \varepsilon_j) \cos^{m-1}\theta_j = 1 \tag{46}$$

By Lemma 3.1 applied in $\mathbb{S}^k$ for any $\varepsilon_0 > 0$ and for all $\phi \in C_0^\infty(\mathbb{S}^k)$ the following inequality holds

$$\left(\int_{\mathbb{S}^k} \phi^2 ds \right)^{1+\frac{2}{k}} \leqslant \left(A_0(k) + \frac{\varepsilon_0}{2} \right) \int_{\mathbb{S}^k} |\nabla \phi|^2 ds \left(\int_{\mathbb{S}^k} |\phi| ds \right)^{\frac{4}{k}},$$

So for a radial function ϕ we obtain

$$\left(\omega_{k-1} \int_0^{\pi/2} \phi^2 \sin^{k-1}\theta d\theta \right)^{1+\frac{2}{k}} \leqslant \left(A_0(k) + \frac{\varepsilon_0}{2} \right) \left(\omega_{k-1} \int_0^{\pi/2} (\phi')^2 \sin^{k-1}\theta d\theta \right)$$

$$\times \left(\omega_{k-1} \int_0^{\pi/2} |\phi| \sin^{k-1}\theta d\theta \right)^{\frac{4}{k}} \tag{47}$$

Let $f \in C_G^\infty$. Then f is a function of one variable and $\eta_j f \in C_G^\infty$ has compact support in $O_{j,\delta}$ which corresponds to the subinterval $[\theta_{j-1}, \theta_j]$. By (47), because of (46), we obtain

$$\left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} (\eta_j f)^2 \sin^{k-1}\theta \cos^{m-1}\theta d\theta \right)^{1+\frac{2}{k}} \leq$$

$$\left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} (\eta_j f)^2 \sin^{k-1}\theta d\theta \right)^{1+\frac{2}{k}} \leq$$

18

$$\left(A_0(k) + \frac{\varepsilon_0}{2}\right) \omega_{m-1}^{-\frac{2}{k}} \left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} ((\eta_j f)')^2 \, sin^{k-1}\theta d\theta\right)$$

$$\times \left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} |\eta_j f| sin^{k-1}\theta d\theta\right)^{\frac{4}{k}} \leq$$

$$\left(A_0(k) + \frac{\varepsilon_0}{2}\right)(1+\varepsilon_j)\omega_{m-1}^{-\frac{2}{k}} \left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} ((\eta_j f)')^2 \, sin^{k-1}\theta cos^{m-1}\theta d\theta\right)$$

$$\times \left(\omega_{k-1}\omega_{m-1} \int_{\theta_{j-1}}^{\theta_j} |\eta_j f| sin^{k-1}\theta cos^{m-1}\theta d\theta\right)^{\frac{4}{k}}$$

or

$$\left(\omega_{k-1}\omega_{m-1} \int_0^{\pi/2} (\eta_j f)^2 sin^{k-1}\theta cos^{m-1}\theta d\theta\right)^{1+\frac{2}{k}} \leq$$

$$\left(A_0(k) + \frac{\varepsilon_0}{2}\right)(1+\varepsilon_j)\omega_{m-1}^{-\frac{2}{k}} \left(\omega_{k-1}\omega_{m-1} \int_0^{\pi/2} ((\eta_j f)')^2 \, sin^{k-1}\theta cos^{m-1}\theta d\theta\right)$$

$$\times \left(\omega_{k-1}\omega_{m-1} \int_0^{\pi/2} |\eta_j f| sin^{k-1}\theta cos^{m-1}\theta d\theta\right)^{\frac{4}{k}}$$

$$(48)$$

Since the covering $(O_{j,\delta})_{j=1,...,J}$ of $\mathbb{S}^n$ depends on $\delta = \delta(\varepsilon)$, we can choose δ such that

$$\left(A_0(k) + \frac{\varepsilon_0}{2}\right)(1 + \max \varepsilon_j) \leq A_0(k) + \varepsilon_0 \qquad (49)$$

and since ε_0 is an arbitrary small positive real we can choose it such that

$$(A_0(k) + \varepsilon_0)\,\omega_{m-1}^{-\frac{2}{k}} \leq A_0(k)\omega_{m-1}^{-\frac{2}{k}} + \varepsilon \qquad (50)$$

By (48), because of (49) and (50), arises

$$\left(\int_{\mathbb{S}^n} (\eta_j f)^2 ds\right)^{1+\frac{2}{k}} \leqslant \left(A_0(k)\omega_{m-1}^{-\frac{2}{k}} + \varepsilon\right) \int_{\mathbb{S}^n} |\nabla(\eta_j f)|^2 ds \left(\int_{\mathbb{S}^n} |\eta_j f|\, ds\right)^{\frac{4}{k}}$$

19

Last inequality means that Lemma 3.1 holds in $\mathbb{S}^n$, and thus by Lemma 3.2 we obtain

$$\left(\int_{\mathbb{S}^n} f^2 ds\right)^{1+\frac{2}{k}} \leqslant \left(A_0(k)\omega_{m-1}^{-\frac{2}{k}} + \varepsilon\right)\left(\int_{\mathbb{S}^n} |\nabla f|^2 ds + B_\varepsilon \int_{\mathbb{S}^n} f^2 ds\right)$$
$$\times \left(\int_{\mathbb{S}^n} |f| ds\right)^{\frac{4}{k}} \tag{51}$$

We have now to prove that the constant $A_0(k)\omega_{m-1}^{-\frac{2}{k}}$ is the best for this inequality. The proof of this part proceeds by contradiction, based on inequality (51). We assume that, for any $\alpha > 0$, there exists $f \in C_0^\infty(\mathbb{S}^n)$ such that

$$\frac{\left(\int_{\mathbb{S}^n} |\nabla f|^2 ds + \alpha \int_{\mathbb{S}^n} f^2 ds\right)\left(\int_{\mathbb{S}^n} |f| ds\right)^{\frac{4}{k}}}{\left(\int_{\mathbb{S}^n} f^2 ds\right)^{1+\frac{2}{k}}} < \frac{1}{A_0(k)\omega_{m-1}^{-\frac{2}{k}}} \tag{52}$$

Using formulas (1.1) and (1.2) in [12] we can write

$$\int_{\mathbb{S}^n} |\nabla f|^2 ds = \omega_{k-1}\omega_{m-1}\int_0^{+\infty} \left|f'(t)(1+t^2)\right|^2 (1+t^2)^{-\frac{k+m}{2}} t^{k-1} dt \tag{53}$$

$$\int_{\mathbb{S}^n} |f| ds = \omega_{k-1}\omega_{m-1}\int_0^{+\infty} |f(t)| (1+t^2)^{-\frac{k+m}{2}} t^{k-1} dt \tag{54}$$

and

$$\int_{\mathbb{S}^n} f^2 ds = \omega_{k-1}\omega_{m-1}\int_0^{+\infty} f^2(t)(1+t^2)^{-\frac{k+m}{2}} t^{k-1} dt \tag{55}$$

For any $\lambda > 0$, define f_λ by $f_\lambda(t) = f(\lambda t)$. So, for any $\lambda > 0$, f_λ has compact support and by (53), (55) and (54) we obtain, respectively

$$\int_{\mathbb{S}^n} |\nabla f|^2 ds = \lambda^{2-k}\omega_{k-1}\omega_{m-1}\int_0^{+\infty}\left|f'(t)\left(1+\frac{t^2}{\lambda^2}\right)\right|^2\left(1+\frac{t^2}{\lambda^2}\right)^{-\frac{k+m}{2}} t^{k-1} dt$$
$$\tag{56}$$

$$\int_{\mathbb{S}^n} |f| ds = \lambda^{-k}\omega_{k-1}\omega_{m-1}\int_0^{+\infty} |f(t)| \left(1+\frac{t^2}{\lambda^2}\right)^{-\frac{k+m}{2}} t^{k-1} dt \tag{57}$$

20

and

$$\int_{\mathbb{S}^n} f^2 ds = \lambda^{-k} \omega_{k-1} \omega_{m-1} \int_0^{+\infty} f^2(t) \left(1 + \frac{t^2}{\lambda^2}\right)^{-\frac{k+m}{2}} t^{k-1} dt \qquad (58)$$

For $\lambda \to +\infty$, by Lebesque's Theorem and because of (56), (58) and (58), inequality (52) yields

$$\frac{\left(\omega_{k-1}\omega_{m-1} \int_0^{+\infty} |f'(t)|^2 t^{k-1} dt\right) \left(\omega_{k-1}\omega_{m-1} \int_0^{+\infty} |f(t)| t^{k-1} dt\right)^{\frac{4}{k}}}{\left(\omega_{k-1}\omega_{m-1} \int_0^{+\infty} f^2(t) t^{k-1} dt\right)^{1+\frac{2}{k}}} < \frac{1}{A_0(k)\omega_{m-1}^{-\frac{2}{k}}}$$

or

$$\frac{\left(\omega_{k-1} \int_0^{+\infty} |f'(t)|^2 t^{k-1} dt\right) \left(\omega_{k-1} \int_0^{+\infty} |f(t)| t^{k-1} dt\right)^{\frac{4}{k}}}{\left(\omega_{k-1} \int_0^{+\infty} f^2(t) t^{k-1} dt\right)^{1+\frac{2}{k}}} < \frac{1}{A_0(k)}$$

$$(59)$$

Since for a radial function f hold

$$f(x) = f(|x|) = f(t) \quad \text{and} \quad |\nabla f(x)| = |f'(t)|$$

we have

$$\int_{\mathbb{R}^k} |\nabla f(x)|^2 \, dx = \omega_{k-1} \int_0^{+\infty} |f'(t)|^2 t^{k-1} dt \qquad (60)$$

Moreover, the following equalities hold

$$\int_{\mathbb{R}^k} |f(x)| \, dx = \omega_{k-1} \int_0^{+\infty} |f(t)| t^{k-1} dt \qquad (61)$$

and

$$\int_{\mathbb{R}^k} f^2(x) dx = \omega_{k-1} \int_0^{+\infty} f^2(t) t^{k-1} dt \qquad (62)$$

By (59), because of (60), (61) and (62), arises

$$\frac{\int_{\mathbb{R}^k} |\nabla f(x)|^2 \, dx \left(\int_{\mathbb{R}^k} |f(x)| \, dx\right)^{\frac{4}{k}}}{\left(\int_{\mathbb{R}^k} f^2(x) dx\right)^{1+\frac{2}{k}}} < \frac{1}{A_0(k)}$$

Last inequality is false (see [2]) and the theorem is proved. $\qquad \square$

21

Proof of Theorem 2.6. The proof of this Theorem is similar to Theorem 2.2. $\square$

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